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Hidden symmetries in (relativistic) hydrodynamics

A C R Mendes, C Neves, W Oliveira and F I Takakura

Departamento de Física, Universidade Federal de Juiz de Fora, 36036-330, Juiz de Fora, MG, Brazil

E-mail: albert@fisica.ufjf.br, cneves@fisica.ufjf.br, wilson@fisica.ufjf.br and takakura@fisica.ufjf.br

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Abstract
In this work, we reveal hidden symmetries on the nonrelativistic and relativistic description of rotational fluid model. Gauge symmetries will be revealed in the context of the symplectic embedding formalism, which allows us to unveil a set of dynamically equivalent hidden symmetries and extra hidden symmetry.

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1. Introduction

Over the last few years, isentropic irrotational fluid mechanics [1] has attracted much attention of theoretical physicists [2, 3]. This model has become a paradigmatic system which drives some theoretical physicists to investigate how some instances of the classical theory are related to D-branes in $(d + 1)$ dimensions and how this relation explains some integrability properties of several models. In [2], the authors demonstrated that the relativistic theory of D-branes is integrable systems by reducing the problem to a d-dimensional nonrelativistic irrotational fluid mechanics. Afterwards, Bazeia and Jackiw [3] found the solutions of this Galileo invariant system in d dimensions that are in connection with the solutions of the relativistic D-brane system in $(d + 1)$ dimensions; in particular, these works clarify the presence of a hidden dynamical Poincaré symmetry of the d-dimensional fluid mechanics. However, this only has validity when the rotational fluid model has a specific potential ($V \propto 1/\rho$ with ρ as being the mass density). This subject is of broad interest since it also offers connections with the hydrodynamical description of quantum mechanics [4, 5], parton model [6], black-hole cosmology [7], hydrodynamics of superfluid systems [8], among others.

It is well known that there is an obstruction to constructing the canonical Lagrangian for rotational system, since the symplectic 2-form does not exist [9]. However, this obstruction problem can be solved by Clebsch parametrization, where new fields are introduced. Recently, two of us proposed [10] a systematic way to give a canonical treatment for rotational system, indeed, the usual result [9, 11] was reproduced and, also, it was suggested that there is a set of equivalent rotational Lagrangian descriptions.

Recently, we proposed a Wess–Zumino (WZ) gauge invariant version for the isentropic irrotational fluid model [12]. In that work, it was also demonstrated that the irrotational fluid model had not a unique WZ version; instead, a set of dynamically WZ gauge invariant versions for this model was obtained. Further, in that paper the extra global symmetries, *Galileo antiboost* and *time rescaling*, first obtained in [3], were lifted to the local one.

In this paper, we propose to discover hidden symmetries on the nonrelativistic isentropic rotational fluid model using the symplectic embedding formalism, which enlarges the phase space with the introduction of WZ fields. We also carry out an investigation of the existent hidden gauge symmetries on the relativistic isentropic rotational fluid model. For this case, an alternative description for the Clebsch decomposition of currents is used [13], with complex potentials taking values in a Kähler manifold. The main advantage of the Kähler parametrization of the fluid current is that it allows a straightforward supersymmetric completion. Plasma physics and heavy-ion collisions, as well as astrophysics and cosmology [14] are some of the applications in the laboratory where relativistic fluid mechanics are very useful. In recent times various extensions and reformulations of the theory have been proposed (see [13], and references therein).

In order to achieve our goal and also to make the paper self-contained, it is organized as follows. In section 2, we present a brief review of the symplectic embedding formalism. In section 3, the nonrelativistic rotational fluid system as well as the obstruction problem will be reviewed. In section 4, some arbitrariness present on the symplectic embedding formalism will be explored in the nonrelativistic rotational model. Further, we will also show that this model has time rescaling symmetry, likely the nonrelativistic irrotational fluid model [3]. In section 5, a brief review of the relativistic rotational fluid will be presented. Afterwards, in subsection 5.2, the relativistic rotational fluid model will be analysed from the symplectic point of view [15, 16]. Here, the Dirac brackets among the fields will be computed. In subsection 5.3, the symplectic embedding formalism [17] will be used and, as a consequence, the global translation symmetry of the velocity potential will be lifted to the local one. The last section will be reserved to stress our conclusion and final discussions.

2. General formalism

In this section, we briefly review the symplectic embedding technique [17] that restores the gauge symmetry. This technique follows the Faddeev–Shatashivilli suggestion [18] and is set up on a contemporary framework to handle constrained models, the symplectic formalism [15, 16].

In order to systematize the symplectic embedding formalism, we consider a general noninvariant mechanical model whose dynamics is governed by a Lagrangian $\mathcal{L}(a_i, \dot{a}_i, t)$ (with $i = 1, 2, \ldots, N$), where a_i and $\dot{a}_i$ are the space and velocity variables, respectively. Note that this model does not result in the loss of generality nor physical content. Following the symplectic method the zeroth-iterative first-order Lagrangian 1-form is written as

$$\mathcal{L}^{(0)}\,\mathrm{d}t = A^{(0)}_\theta\,\mathrm{d}\xi^{(0)\theta} - V^{(0)}(\xi)\,\mathrm{d}t. \tag{1}$$

The symplectic variables are

$$\xi^{(0)\theta} = \begin{cases} a_i, & \text{with } \theta = 1, 2, \ldots, N \\ p_i, & \text{with } \theta = N+1, N+2, \ldots, 2N, \end{cases} \tag{2}$$

and $A_\theta^{(0)}$ are the canonical momenta and $V^{(0)}$ is the symplectic potential. From the Euler–Lagrange equations of motion, the symplectic tensor is obtained as

$$f_{\theta\beta}^{(0)} = \frac{\partial A_\beta^{(0)}}{\partial \xi^{(0)\theta}} - \frac{\partial A_\theta^{(0)}}{\partial \xi^{(0)\beta}}. \tag{3}$$

When the 2-form $f \equiv \frac{1}{2} f_{\theta\beta}\, \mathrm{d}\xi^\theta \wedge \mathrm{d}\xi^\beta$ is singular, the symplectic matrix (3) has a zero mode $(\nu^{(0)})$ that generates a new constraint when contracted with the gradient of the symplectic potential,

$$\Omega^{(0)} = \nu^{(0)\theta} \frac{\partial V^{(0)}}{\partial \xi^{(0)\theta}}. \tag{4}$$

This constraint is introduced into the zeroth-iterative Lagrangian 1-form, equation (1), through a Lagrange multiplier η, generating the next one

$$\begin{aligned} \mathcal{L}^{(1)}\, \mathrm{d}t &= A_\theta^{(0)}\, \mathrm{d}\xi^{(0)\theta} + \mathrm{d}\eta \Omega^{(0)} - V^{(0)}(\xi)\, \mathrm{d}t, \\ &= A_\gamma^{(1)}\, \mathrm{d}\xi^{(1)\gamma} - V^{(1)}(\xi)\, \mathrm{d}t, \end{aligned} \tag{5}$$

with $\gamma = 1, 2, \ldots, (2N+1)$ and

$$V^{(1)} = V^{(0)}|_{\Omega^{(0)}=0}, \qquad \xi^{(1)_\gamma} = (\xi^{(0)\theta}, \eta), \qquad A_\gamma^{(1)} = \left(A_\theta^{(0)}, \Omega^{(0)}\right). \tag{6}$$

As a consequence, the first-iterative symplectic tensor is computed as

$$f_{\gamma\beta}^{(1)} = \frac{\partial A_\beta^{(1)}}{\partial \xi^{(1)\gamma}} - \frac{\partial A_\gamma^{(1)}}{\partial \xi^{(1)\beta}}. \tag{7}$$

If this tensor is nonsingular, the iterative process stops and the Dirac brackets among the phase-space variables are obtained from the inverse matrix $\left(f_{\gamma\beta}^{(1)}\right)^{-1}$ and, consequently, the Hamilton equation of motion can be computed and solved, as discussed in [19]. It is well known that a physical system can be described at least classically in terms of a symplectic manifold M. From a physical point of view, M is the phase space of the system while a nondegenerate closed 2-form f can be identified as being the Poisson bracket. The dynamics of the system is determined by just specifying a real-valued function (Hamiltonian) H on the phase space, i.e., one of these real-valued functions solves the Hamilton equation, namely,

$$\iota(X) f = \mathrm{d}H, \tag{8}$$

and the classical dynamical trajectories of the system in the phase space are obtained. It is important to mention that if f is nondegenerate, equation (8) has a unique solution. The nondegeneracy of f means that the linear map $\flat : TM \to T^*M$ defined by $\flat(X) := \flat(X) f$ is an isomorphism; due to this equation (8) is solved uniquely for any Hamiltonian $(X = \flat^{-1}(\mathrm{d}H))$. In contrast, the tensor has a zero mode and a new constraint arises, indicating that the iterative process goes on until the symplectic matrix becomes nonsingular or singular. If this matrix is nonsingular, the Dirac brackets will be determined. In [19], the authors consider in detail the case when f is degenerate, which usually arises when constraints are presented on the system. In which case, (M, f) is called the presymplectic manifold. As a consequence, the Hamilton equation (8) may or may not possess solutions, or possess nonunique solutions. Conversely, if this matrix is singular and the respective zero mode does not generate a new constraint, the system has a symmetry.

After this brief introduction, the symplectic embedding formalism will be systematized. The main idea of this embedding formalism is to introduce extra fields into the model in order to obstruct the solutions of the Hamiltonian equations of motion. It begins with the introduction of two arbitrary functions dependent on the original phase space and the WZ variable, namely $\Psi(a_i, p_i)$ and $G(a_i, p_i, \eta)$ into the first-order Lagrangian 1-form as follows:

$$\tilde{\mathcal{L}}^{(0)}\,\mathrm{d}t = A^{(0)}_{\theta}\,\mathrm{d}\xi^{(0)\theta} + \Psi\,\mathrm{d}\eta - \tilde{V}^{(0)}(\xi)\,\mathrm{d}t, \tag{9}$$

with

$$\tilde{V}^{(0)} = V^{(0)} + G(a_i, p_i, \eta), \tag{10}$$

where the arbitrary function $G(a_i, p_i, \eta)$ is expressed as an expansion in terms of the WZ field, given by

$$G(a_i, p_i, \eta) = \sum_{n=1}^{\infty} \mathcal{G}^{(n)}(a_i, p_i, \eta), \qquad \mathcal{G}^{(n)}(a_i, p_i, \eta) \sim \eta^n \tag{11}$$

and satisfies the following boundary condition:

$$G(a_i, p_i, \eta = 0) = 0. \tag{12}$$

The symplectic variables were extended to also contain the WZ variable $\tilde{\xi}^{(0)\tilde{\theta}} = (\xi^{(0)\theta}, \eta)$ (with $\tilde{\theta} = 1, 2, \ldots, 2N+1$) and the first-iterative symplectic potential becomes

$$\tilde{V}^{(0)}(a_i, p_i, \eta) = V^{(0)}(a_i, p_i) + \sum_{n=1}^{\infty} \mathcal{G}^{(n)}(a_i, p_i, \eta). \tag{13}$$

In this context, the canonical momenta are

$$\tilde{A}^{(0)}_{\tilde{\theta}} = \begin{cases} A^{(0)}_{\theta}, & \text{with } \tilde{\theta} = 1, 2, \ldots, 2N \\ \Psi, & \text{with } \tilde{\theta} = 2N+1 \end{cases} \tag{14}$$

and the new symplectic tensor, given by

$$\tilde{f}^{(0)}_{\tilde{\theta}\tilde{\beta}} = \frac{\partial \tilde{A}^{(0)}_{\tilde{\beta}}}{\partial \tilde{\xi}^{(0)\tilde{\theta}}} - \frac{\partial \tilde{A}^{(0)}_{\tilde{\theta}}}{\partial \tilde{\xi}^{(0)\tilde{\beta}}}, \tag{15}$$

is

$$\tilde{f}^{(0)}_{\tilde{\theta}\tilde{\beta}} = \begin{pmatrix} f^{(0)}_{\theta\beta} & f^{(0)}_{\theta\eta} \\ f^{(0)}_{\eta\beta} & 0 \end{pmatrix}. \tag{16}$$

The implementation of the symplectic embedding scheme follows with two steps: the first one is addressed to compute $\Psi(a_i, p_i)$ while the second one is dedicated to the calculation of $G(a_i, p_i, \eta)$. In order to begin with the first step, we impose that this new symplectic tensor $(\tilde{f}^{(0)})$ has a zero mode $\tilde{\nu}$; consequently, we get the following condition:

$$\tilde{\nu}^{(0)\tilde{\theta}}\,\tilde{f}^{(0)}_{\tilde{\theta}\tilde{\beta}} = 0. \tag{17}$$

Note that, at this point, f becomes degenerate and, in consequence, we introduce an obstruction to solve, in an unique way, the Hamilton equation of motion given in equation (8). Assuming that the zero mode $\tilde{\nu}^{(0)\tilde{\theta}}$ is

$$\tilde{\nu}^{(0)} = (\mu^{\theta} \quad 1), \tag{18}$$

and using the relation given in equation (17) together with equation (16), we get a set of equations, namely,

$$\mu^{\theta} f^{(0)}_{\theta\beta} + f^{(0)}_{\eta\beta} = 0, \tag{19}$$

where

$$f^{(0)}_{\eta\beta} = \frac{\partial A^{(0)}_{\beta}}{\partial \eta} - \frac{\partial \Psi}{\partial \xi^{(0)\beta}}. \tag{20}$$

Observe that the matrix elements μ^θ are chosen in order to disclose a desired gauge symmetry. Note that in this formalism the zero mode $\tilde{\nu}^{(0)\tilde{\theta}}$ is the gauge symmetry generator. At this point, it is deserved to mention that this characteristic is important because it opens up the possibility of disclosing the desired hidden gauge symmetry from the noninvariant model. It provides the symplectic embedding formalism with some power to deal with noninvariant systems. From the relation given in equation (17) some differential equations involving $\Psi(a_i, p_i)$ are obtained, equation (19), and after a straightforward computation, $\Psi(a_i, p_i)$ can be determined.

In order to compute $G(a_i, p_i, \eta)$ in the second step, we impose that no more constraints arise from the contraction of the zero mode ($\tilde{\nu}^{(0)\tilde{\theta}}$) with the gradient of the potential $\tilde{V}^{(0)}(a_i, p_i, \eta)$. This condition generates a general differential equation, which reads as

$$\begin{aligned} 0 &= \tilde{\nu}^{(0)\tilde{\theta}} \frac{\partial \tilde{V}^{(0)}(a_i, p_i, \eta)}{\partial \tilde{\xi}^{(0)\tilde{\theta}}} \\ &= \mu^\theta \frac{\partial V^{(0)}(a_i, p_i)}{\partial \xi^{(0)\theta}} + \mu^\theta \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \eta)}{\partial \xi^{(0)\theta}} + \mu^\theta \frac{\partial \mathcal{G}^{(2)}(a_i, p_i, \eta)}{\partial \xi^{(0)\theta}} + \cdots \\ &\quad + \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \eta)}{\partial \eta} + \frac{\partial \mathcal{G}^{(2)}(a_i, p_i, \eta)}{\partial \eta} + \cdots, \end{aligned} \tag{21}$$

that allows us to compute all correction terms $\mathcal{G}^{(n)}(a_i, p_i, \eta)$ in the order of η. Note that this polynomial expansion in terms of η is equal to zero; subsequently, whole coefficients for each order in η must be null identically. In view of this, each correction term in the order of η is determined. For a linear correction term, we have

$$\mu^\theta \frac{\partial V^{(0)}(a_i, p_i)}{\partial \xi^{(0)\theta}} + \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \eta)}{\partial \eta} = 0. \tag{22}$$

For a quadratic correction term, we get

$$\mu^\theta \frac{\partial \mathcal{G}^{(1)}(a_i, p_i, \eta)}{\partial \xi^{(0)\theta}} + \frac{\partial \mathcal{G}^{(2)}(a_i, p_i, \eta)}{\partial \eta} = 0. \tag{23}$$

From these equations, a recursive equation for $n \geqslant 2$ is proposed as

$$\mu^\theta \frac{\partial \mathcal{G}^{(n-1)}(a_i, p_i, \eta)}{\partial \xi^{(0)\theta}} + \frac{\partial \mathcal{G}^{(n)}(a_i, p_i, \eta)}{\partial \eta} = 0, \tag{24}$$

that allows us to compute the remaining correction terms in the order of η. This iterative process is successively repeated until equation (21) becomes identically null; consequently, the extra term $G(a_i, p_i, \eta)$ is obtained explicitly. Then, the gauge invariant Hamiltonian, identified as being the symplectic potential, is obtained as

$$\tilde{\mathcal{H}}(a_i, p_i, \eta) = V^{(0)}(a_i, p_i) + G(a_i, p_i, \eta), \tag{25}$$

and the zero mode $\tilde{\nu}^{(0)\tilde{\theta}}$ is identified as being the generator of an infinitesimal gauge transformation, given by

$$\delta \tilde{\xi}^{\tilde{\theta}} = \varepsilon \tilde{\nu}^{(0)\tilde{\theta}}, \tag{26}$$

where ε is an infinitesimal parameter.

3. Rotational fluid mechanics

In this section, the obstruction problem to construct both a canonical formalism and the Lagrangian description for the rotational fluid mechanics will be described. Let us consider a inviscid, isentropic and compressible fluid, whose dynamics is governed by the continuity and Euler equations, which are read as

$$\begin{aligned} &\frac{\partial \rho(t,\vec{r})}{\partial t} + \nabla \cdot (\rho(t,\vec{r}) \cdot \vec{v}(t,\vec{r})) = 0, \\ &\frac{\partial \vec{v}(t,\vec{r})}{\partial t} + \vec{v}(t,\vec{r}) \cdot \nabla \vec{v}(t,\vec{r}) = \vec{f}(t,\vec{r}), \end{aligned} \tag{27}$$

where $\rho(t,\vec{r})$ and $\vec{v}(t,\vec{r})$ denote the mass density and the velocity field, respectively. Here, $\rho(t,\vec{r})\vec{v}(t,\vec{r})$ is the current and $\vec{f}(t,\vec{r}))$ is the force, which will be kept arbitrary for the time being.

It is well known that a dynamical system is most powerful presented from a canonical formulation. Due to this, it is important to remark that the equations, given in equation (27), can be obtained by Poisson-bracketing the fields $\rho(t,\vec{r})$ and $\vec{v}(t,\vec{r})$ with the following Hamiltonian,

$$\mathcal{H} = \tfrac{1}{2}\rho v^2 + V(\rho), \tag{28}$$

with $V(\rho)$ being an interactive potential. As a consequence, the Hamilton equations of motion are

$$\frac{\partial \rho(t,\vec{r})}{\partial t} = \{H, \rho(t,\vec{r})\}, \qquad \frac{\partial \vec{v}(t,\vec{r})}{\partial t} = \{H, \vec{v}(t,\vec{r})\}, \tag{29}$$

provided that the nonvanishing Poisson brackets among the fields must be taken as

$$\{v^i(\vec{r}), \rho(\vec{r}\,')\} = \frac{\partial \delta(\vec{r} - \vec{r}\,')}{\partial x_i}, \qquad \{v^i(\vec{r}), v^j((\vec{r})\} = -\frac{\omega_{ij}(\vec{r},\vec{r}\,')}{\rho(\vec{r})}\delta(\vec{r} - \vec{r}\,'), \tag{30}$$

where the vorticity $\vec{w}$ is

$$\omega_{ij}(\vec{r},\vec{r}\,') = \frac{\partial v^j(\vec{r}\,')}{\partial x^i} - \frac{\partial v^i(\vec{r})}{\partial x'^j}. \tag{31}$$

Now, we shall discuss the symplectic canonical formulation of the rotational fluid mechanics, which is an attempt to simplify the usual canonical process. This symplectic process was briefly presented in section 2 where the fundamental brackets among the fields were postulated as being the inverse of the symplectic 2-form f^{ij}, which reads as

$$f^{ij} = \begin{pmatrix} 0 & -\dfrac{\partial \delta(\vec{r} - \vec{r}\,')}{\partial x_j} \\ \dfrac{\partial \delta(\vec{r} - \vec{r}\,')}{\partial x_i} & -\dfrac{\omega_{ij}(\vec{r},\vec{r}\,')}{\rho(\vec{r})}\delta(\vec{r} - \vec{r}\,') \end{pmatrix}. \tag{32}$$

Note that this matrix is singular and has the following zero mode,

$$\nu(\vec{r}\,') = \begin{pmatrix} 0 & \dfrac{\partial C}{\partial v^i(\vec{r}\,')} \end{pmatrix}, \qquad \text{with} \qquad C = \int \mathrm{d}\vec{r}\,'' \left(\epsilon_{kmn} v^k(\vec{r}\,'') \frac{\partial v^n(\vec{r}\,'')}{\partial x''_m}\right). \tag{33}$$

In fact,

$$\begin{aligned} \int \mathrm{d}\vec{r}\,' \frac{\partial C}{\partial v^i(\vec{r}\,')} \omega_{ij}(\vec{r},\vec{r}\,')\delta(\vec{r} - \vec{r}\,') &= \int \mathrm{d}\vec{r}\,'' \left[\epsilon_{kmn}\left(\delta^{ki}\delta(\vec{r}\,'' - \vec{r})\frac{\partial v^n(\vec{r}\,'')}{\partial x''_m}\right.\right. \\ &\quad \left.\left. +\ v^k(\vec{r}\,'')\frac{\partial(\delta^{ni}\delta(\vec{r}\,'' - \vec{r}))}{\partial x''_m}\right)\left(\frac{\partial v^j(\vec{r})}{\partial x^i} - \frac{\partial v^i(\vec{r})}{\partial x^j}\right)\right] \\ &= \epsilon_{imn}\frac{\partial v^n(\vec{r})}{\partial x_m}\left(\frac{\partial v^j(\vec{r})}{\partial x^i} - \frac{\partial v^i(\vec{r})}{\partial x^j}\right) - \epsilon_{kmi}\frac{\partial v^k(\vec{r})}{\partial x_m}\left(\frac{\partial v^j(\vec{r})}{\partial x^i} - \frac{\partial v^i(\vec{r})}{\partial x^j}\right) \\ &= 2\epsilon_{imn}\frac{\partial v^n(\vec{r})}{\partial x_m}\left(\frac{\partial v^j(\vec{r})}{\partial x^i} - \frac{\partial v^i(\vec{r})}{\partial x^j}\right), \\ &= 0, \end{aligned} \tag{34}$$

since ϵ_{imn} is an antisymmetric tensor. In consequence, f^{ij} has no inverse and, then, the symplectic 2-form f_{ij} does not exist. Therefore, the existence of such a constant C creates an obstruction in the inversion of the symplectic matrix and, as a consequence, a canonical Lagrangian formulation for rotational fluid mechanics is lacking. To overcome this kind of problem and then neutralize the obstruction, the Clebsch parametrization process is usually implemented. To this end, the velocity vector field becomes

$$\vec{v} = \vec{\nabla}\theta + \alpha\vec{\nabla}\beta, \tag{35}$$

with three suitable chosen scalar functions θ, α e β, where (α, β) are called 'Gaussian potentials' [20]. In this parametrization, the vorticity reads

$$\vec{\omega} = \vec{\nabla}\alpha \times \vec{\nabla}\beta, \tag{36}$$

and the Lagrangian is taken as

$$\mathcal{L} = -\rho(\dot{\theta} + \alpha\dot{\beta}) - \tfrac{1}{2}\rho(\vec{\nabla}\theta + \alpha\vec{\nabla}\beta)^2 - V(\rho), \tag{37}$$

where $V(\rho)$ is an arbitrary interaction potential. This result was also obtained by two of us by using an alternative formulation in [10]. In view of this, we argue in this work [10] that there is not a unique solution to implement the Clebsch parametrization for the rotational fluid model, i.e., there is another way to carry out the Clebsch parametrization in order to solve the obstruction and, then, to construct the Lagrangian description.

4. WZ gauge invariant rotational fluid mechanics

Recently, we have investigated the WZ hidden symmetry on the irrotational fluid model [12] in the symplectic embedding framework. In this work, we have demonstrated that the irrotational fluid model has a set of dynamically gauge invariant Lagrangian descriptions and, further, we also lift the global extra symmetries [3] in the local one. In seems important since, in some way, the original gauge invariant nature of the original theory, the D-brane in $(d+1)$ dimensions, is lost after the reduction process to a d-dimensional fluid model [2].

In the present section, we will investigate the presence of hidden symmetries on the nonrelativistic rotational fluid model and, as well as the time rescaling symmetry, by using the symplectic embedding formalism [17]. As discussed in the last section, this model presents an obstruction to constructing the canonical Lagrangian formulation, which is solved by using Clebsch parametrization [9–11]. Hence, a Lagrangian description for the rotational fluid is proposed, where two new fields are introduced, namely, the 'Gaussian potentials' (α, β). However, this Lagrangian does not display a gauge symmetry due to, in the Diracs language, the presence of second class constraints. Due to some advantage in avoiding some problems, for example, operator ordering problem at the quantum level and the identification of an innate interaction on the system, it seems more adequate to present a gauge invariant description for a model. In order to fill this lack, we propose to obtain a gauge invariant Lagrangian for the rotational fluid. To this end, we begin with the usual second class Lagrangian for the rotational fluid model and apply the symplectic embedding formalism, where the phase space will be extended with the introduction of the WZ fields. To start with, we change the Lagrangian (37) introducing two arbitrary functions $\Psi \equiv \Psi(\rho, \theta, \alpha, \beta)$ and $G \equiv G(\rho, \theta, \alpha, \beta, \eta)$, namely,

$$\tilde{\mathcal{L}} = -\rho(\dot{\theta} + \alpha\dot{\beta}) + \Psi\dot{\eta} - \tfrac{1}{2}\rho(\vec{\nabla}\theta + \alpha\vec{\nabla}\beta)^2 - V(\rho) - G, \tag{38}$$

where G is a function expressed as

$$G(\rho, \theta, \alpha, \beta, \eta) = \sum_{n=1}^{\infty} \mathcal{G}^n \qquad \text{with} \qquad \mathcal{G}^n \propto \eta^n, \tag{39}$$

and satisfies the following boundary condition,

$$G(\rho,\theta,\alpha,\beta,\eta=0)=0. \tag{40}$$

The extended symplectic field and the corresponding singular matrix are

$$\tilde{\xi}^{(0)} \equiv (\rho,\theta,\alpha,\beta,\eta),$$

$$\tilde{f}^{(0)} = \begin{pmatrix} 0 & -\delta^{(d)}(\vec{x}-\vec{y}) & 0 & -\alpha(x)\delta^{(d)}(\vec{x}-\vec{y}) & \dfrac{\delta\Psi(\vec{x})}{\delta\rho(\vec{y})} \\ \delta^{(d)}(\vec{x}-\vec{y}) & 0 & 0 & 0 & \dfrac{\delta\Psi(\vec{x})}{\delta\theta(\vec{y})} \\ 0 & 0 & 0 & -\rho(x)\delta^{(d)}(\vec{x}-\vec{y}) & \dfrac{\delta\Psi(\vec{x})}{\delta\alpha(\vec{y})} \\ \alpha(y)\delta^{(d)}(\vec{x}-\vec{y}) & 0 & \rho(y)\delta^{(d)}(\vec{x}-\vec{y}) & 0 & \dfrac{\delta\Psi(\vec{x})}{\delta\beta(\vec{y})} \\ -\dfrac{\delta\Psi(\vec{y})}{\delta\rho(\vec{x})} & -\dfrac{\delta\Psi(\vec{y})}{\delta\theta(\vec{x})} & -\dfrac{\delta\Psi(\vec{y})}{\delta\alpha(\vec{x})} & -\dfrac{\delta\Psi(\vec{y})}{\delta\beta(\vec{x})} & 0 \end{pmatrix}. \tag{41}$$

In order to investigate hidden symmetries on the model, we start considering the following zero mode:

$$\nu = (0 \quad 1 \quad 0 \quad 0 \quad -1). \tag{42}$$

Following the symplectic embedding process, the contraction of this zero mode with the symplectic matrix, equation (41), generates a set of differential equations, namely,

$$\begin{gathered} \int \mathrm{d}\vec{x}\left(\delta^{(d)}(\vec{x}-\vec{y}) + \frac{\delta\Psi(\vec{y})}{\delta\rho(\vec{x})}\right) = 0, \\ \int \mathrm{d}\vec{x}\left(\frac{\delta\Psi(\vec{y})}{\delta\theta(\vec{x})}\right) = 0, \qquad \int \mathrm{d}\vec{x}\left(\frac{\delta\Psi(\vec{y})}{\delta\alpha(\vec{x})}\right) = 0, \\ \int \mathrm{d}\vec{x}\left(\frac{\delta\Psi(\vec{y})}{\delta\beta(\vec{x})}\right) = 0, \qquad \int \mathrm{d}\vec{x}\left(\frac{\delta\Psi(\vec{y})}{\delta\beta(\vec{x})}\right) = 0, \end{gathered} \tag{43}$$

which after a calculation gives

$$\Psi = -\rho. \tag{44}$$

Hence, the Lagrangian given in equation (38) becomes

$$\tilde{\mathcal{L}} = -\rho(\dot{\theta}+\alpha\dot{\beta}) - \rho\dot{\eta} - \tfrac{1}{2}\rho(\vec{\nabla}\theta+\alpha\vec{\nabla}\beta)^2 - V(\rho) - G. \tag{45}$$

Now, the second step of the symplectic embedding formalism begins. To compute the function G, the contraction of the zero mode with the gradient of the symplectic potential must be null. Due to this, we get the following relation,

$$\int \mathrm{d}\vec{x}\left(\rho(\vec{\nabla}\theta+\alpha\vec{\nabla}\beta)\vec{\nabla}\delta^{(d)}(\vec{x}-\vec{y}) + \sum_{n=1}^{\infty}\frac{\delta\mathcal{G}^{(n)}(x)}{\delta\theta(y)} - \sum_{n=1}^{\infty}\frac{\delta\mathcal{G}^{(n)}(x)}{\delta\eta(y)}\right) = 0, \tag{46}$$

which allows us to compute all correction terms in the order of η. For linear terms, we have,

$$\int \mathrm{d}\vec{x}\left(\rho(\vec{\nabla}\theta+\alpha\vec{\nabla}\beta)\vec{\nabla}\delta^{(d)}(\vec{x}-\vec{y}) - \frac{\delta\mathcal{G}^{(1)}(x)}{\delta\eta(y)}\right) = 0, \tag{47}$$

which after a calculation gives

$$\mathcal{G}^{(1)} = \rho(\vec{\nabla}\theta+\alpha\vec{\nabla}\beta)\vec{\nabla}\eta. \tag{48}$$

For a quadratic correction term, we have

$$\int \mathrm{d}\vec{x}\left(\rho\vec{\nabla}\eta\vec{\nabla}\delta^{(d)}(\vec{x}-\vec{y})-\frac{\delta\mathcal{G}^{(2)}(x)}{\delta\eta(y)}\right)=0, \tag{49}$$

which generates the following solution:

$$\mathcal{G}^{(2)}=\tfrac{1}{2}\rho(\vec{\nabla}\eta)^2. \tag{50}$$

As $\mathcal{G}^{(2)}$ has no dependence on θ, the remaining correction terms are null, i.e., $\mathcal{G}^{(n)}=0$ for $n\geqslant 3$. Hence, the Lagrangian is written as

$$\tilde{\mathcal{L}}=-\rho(\dot{\theta}+\alpha\dot{\beta})-\rho\dot{\eta}-\tfrac{1}{2}\rho(\vec{\nabla}\theta+\alpha\vec{\nabla}\beta)^2-V(\rho)-\rho(\vec{\nabla}\theta+\alpha\vec{\nabla}\beta)\vec{\nabla}\eta-\tfrac{1}{2}\rho(\vec{\nabla}\eta)^2, \tag{51}$$

which is invariant under the following infinitesimal transformation,

$$\delta\rho=0,\qquad \delta\theta=\varepsilon,\qquad \delta\alpha=0,\qquad \delta\beta=0,\qquad \delta\eta=-\varepsilon, \tag{52}$$

where ε is a time-dependent parameter.

At this point, it is interesting to remember that in [17] it was established that WZ symmetries are not unique; instead, they belong to a family of dynamically equivalent WZ symmetries. Indeed, this was demonstrated by three of us in [12] in the context of the nonrelativistic irrotational fluid model.

Now, we are interested in revealing the local time rescaling symmetry. To this end, the following zero mode, read as

$$\nu=(-\rho\quad\theta\quad\alpha\quad 0\quad -1), \tag{53}$$

is proposed. Contracting this zero mode with the symplectic matrix above, a set of differential equations is obtained as

$$\begin{gathered}\int \mathrm{d}\vec{x}\left(\theta(y)\delta^{(d)}(\vec{x}-\vec{y})+\frac{\delta\Psi(\vec{y})}{\delta\rho(\vec{x})}\right)=0,\\ \int \mathrm{d}\vec{x}\left(+\rho(y)\delta^{(d)}(\vec{x}-\vec{y})+\frac{\delta\Psi(\vec{y})}{\delta\theta(\vec{x})}\right)=0,\\ \int \mathrm{d}\vec{x}\left(\frac{\delta\Psi(\vec{y})}{\delta\alpha(\vec{x})}\right)=0,\qquad \int \mathrm{d}\vec{x}\left(\frac{\delta\Psi(\vec{y})}{\delta\beta(\vec{x})}\right)=0,\\ \int \mathrm{d}\vec{x}\left(-\rho(y)\frac{\delta\Psi(\vec{y})}{\delta\rho(\vec{x})}+\theta(y)\frac{\delta\Psi(\vec{y})}{\delta\theta(\vec{x})}+\alpha(y)\frac{\delta\Psi(\vec{y})}{\delta\alpha(\vec{x})}\right)=0.\end{gathered} \tag{54}$$

After a straightforward computation, we get

$$\Psi=-\theta\rho. \tag{55}$$

Then the Lagrangian becomes

$$\tilde{\mathcal{L}}=-\rho(\dot{\theta}+\alpha\dot{\beta})-(\theta\rho)\dot{\eta}-\tfrac{1}{2}\rho(\vec{\nabla}\theta+\alpha\vec{\nabla}\beta)^2-V(\rho)-G. \tag{56}$$

After this point, we begin with the second step of the symplectic embedding formalism. To this end, we impose that the contraction of the zero mode, equation (53), with the gradient of the symplectic potential generates an identically null result, namely,

$$\int \mathrm{d}\vec{x}\;\nu^i(y)\frac{\delta\tilde{V}(x)}{\delta\xi^i(y)}=0, \tag{57}$$

where $\tilde{V}(x) = \frac{1}{2}\rho(\vec{\bigtriangledown}\theta + \alpha\vec{\bigtriangledown}\beta)^2 + V(\rho) + G(x)$. From this condition, the following general differential equation is obtained,

$$\begin{aligned} 0 = \int \mathrm{d}\vec{x} \Bigg(& -\frac{1}{2}\rho(y)\delta^{(d)}(\vec{x}-\vec{y})(\vec{\bigtriangledown}\theta(x) + \alpha(x)\vec{\bigtriangledown}\beta(x))^2 - \rho(y)\frac{\delta V(\rho)}{\delta\rho(y)} \\ & + \theta(y)\rho(x)(\vec{\bigtriangledown}\theta(x) + \alpha(x)\vec{\bigtriangledown}\beta(x)) \cdot \vec{\bigtriangledown}\delta^{(d)}(\vec{x}-\vec{y}) \\ & + \alpha(y)\rho(x)(\vec{\bigtriangledown}\theta(x) + \alpha(x)\vec{\bigtriangledown}\beta(x)) \cdot \vec{\bigtriangledown}\beta(x)\delta^{(d)}(\vec{x}-\vec{y}) - \rho(y)\sum_{n=1}^{\infty}\frac{\delta\mathcal{G}^{(n)}(x)}{\delta\rho(y)} \\ & + \theta\sum_{n=1}^{\infty}\frac{\delta\mathcal{G}^{(n)}(x)}{\delta\theta(y)} + \alpha\sum_{n=1}^{\infty}\frac{\delta\mathcal{G}^{(n)}(x)}{\delta\alpha(y)} - \sum_{n=1}^{\infty}\frac{\delta\mathcal{G}^{(n)}(x)}{\delta\eta(y)}\Bigg), \end{aligned} \tag{58}$$

where the relation given in equation (39) was used. This allows the computation of the whole correction terms in the order of η. For the linear correction term ($\mathcal{G}^{(1)}(x)$), we get

$$\begin{aligned} 0 = \int \mathrm{d}\vec{x} \Bigg(& -\frac{1}{2}\rho(y)\delta^{(d)}(\vec{x}-\vec{y})(\vec{\bigtriangledown}\theta(x) + \alpha(x)\vec{\bigtriangledown}\beta(x))^2 - \rho(y)\frac{\delta V(\rho)}{\delta\rho(y)} \\ & + \theta(y)\rho(x)(\vec{\bigtriangledown}\theta(x) + \alpha(x)\vec{\bigtriangledown}\beta(x)) \cdot \vec{\bigtriangledown}\delta^{(d)}(\vec{x}-\vec{y}) \\ & + \alpha(y)\rho(x)(\vec{\bigtriangledown}\theta(x) + \alpha(x)\vec{\bigtriangledown}\beta(x)) \cdot \vec{\bigtriangledown}\beta(x)\delta^{(d)}(\vec{x}-\vec{y}) - \frac{\delta\mathcal{G}^{(1)}(x)}{\delta\eta(y)}\Bigg), \end{aligned} \tag{59}$$

which after a computation gives

$$\mathcal{G}^{(1)} = \frac{1}{2}\rho\eta(\vec{\bigtriangledown}\theta + \alpha\vec{\bigtriangledown}\beta)^2 - \eta\rho\frac{\delta}{\delta\rho(x)}\int \mathrm{d}\vec{z}\,V(\rho). \tag{60}$$

For the quadratic correction term, we have

$$\int \mathrm{d}\vec{x}\left(-\rho(y)\frac{\mathcal{G}^{(1)}(x)}{\delta\rho(y)} + \theta(y)\frac{\mathcal{G}^{(1)}(x)}{\delta\theta(y)} + \alpha(y)\frac{\mathcal{G}^{(1)}(x)}{\delta\alpha(y)} - \frac{\mathcal{G}^{(2)}(x)}{\delta\eta(y)}\right) = 0, \tag{61}$$

with the following solution,

$$\mathcal{G}^{(2)} = \frac{1}{4}\rho\eta^2(\vec{\bigtriangledown}\theta + \alpha\vec{\bigtriangledown}\beta)^2 + \frac{1}{2}\eta\rho\frac{\delta}{\delta\rho(x)}\left(\int \mathrm{d}\vec{y}\,\eta\rho\frac{\delta}{\delta\rho(y)}\int \mathrm{d}\vec{z}\,V(\rho)\right). \tag{62}$$

The next correction term is

$$\mathcal{G}^{(3)} = +\frac{1}{12}\rho\eta^3(\vec{\bigtriangledown}\theta + \alpha\vec{\bigtriangledown}\beta)^2 - \frac{1}{6}\eta\rho\frac{\delta}{\delta\rho(x)}\left(\int \mathrm{d}\vec{w}\,\eta\rho\frac{\delta}{\delta\rho(x)}\left(\int \mathrm{d}\vec{w}\,\eta\rho\frac{\delta}{\delta\rho(y)}\int \mathrm{d}\vec{z}\,V(\rho)\right)\right). \tag{63}$$

As the last correction term depends on ρ only, there is a recursive formula for the remaining terms in the order of η, which reads as

$$\begin{aligned} \mathcal{G}^{(n)} = {} & \frac{1}{2n!}\rho\eta^n(\vec{\bigtriangledown}\theta + \alpha\vec{\bigtriangledown}\beta)^2 + (-1)^n\frac{1}{n!}\eta\rho\frac{\delta}{\delta\rho(x_n)}\left(\int \mathrm{d}\vec{x}_{(n-1)}\,\eta\rho\frac{\delta}{\delta\rho(x_{(n-1)})}\right. \\ & \left.\times\left(\ldots\left(\int \mathrm{d}\vec{x}_2\,\eta\rho\frac{\delta}{\delta\rho(x_2)}\left(\int \mathrm{d}\vec{x}_1\,\eta\rho\frac{\delta}{\delta\rho(x_1)}\int \mathrm{d}\vec{z}\,V(\rho)\right)\right)\ldots\right)\right). \end{aligned} \tag{64}$$

Hence, the gauge invariant Lagrangian for the rotational fluid model is

$$\begin{aligned} \tilde{\mathcal{L}} = {} & -\rho(\dot{\theta} + \alpha\dot{\beta}) - (\theta\rho)\dot{\eta} - \frac{1}{2}\rho(\vec{\bigtriangledown}\theta + \alpha\vec{\bigtriangledown}\beta)^2 e^n - V(\rho) - (-1)^n\frac{1}{n!}\eta\rho\frac{\delta}{\delta\rho(x_n)} \\ & \times\left(\int \mathrm{d}\vec{x}_{(n-1)}\eta\rho\frac{\delta}{\delta\rho(x_{(n-1)})}\left(\ldots\left(\int \mathrm{d}\vec{x}_2\eta\rho\frac{\delta}{\delta\rho(x_2)}\left(\int \mathrm{d}\vec{x}_1\eta\rho\frac{\delta}{\delta\rho(x_1)}\int \mathrm{d}\vec{z}\,V(\rho)\right)\right)\ldots\right)\right). \end{aligned} \tag{65}$$

As the hidden symmetries [2, 3] have validated only a special interactive potential, namely $V(\rho) = g/\rho$, the above Lagrangian becomes

$$\tilde{\mathcal{L}} = -\rho(\dot{\theta} + \alpha\dot{\beta}) - (\theta\rho)\dot{\eta} - \frac{1}{2}\rho(\vec{\nabla}\theta + \alpha\vec{\nabla}\beta)^2\,\mathrm{e}^n - \frac{g}{\rho}\,\mathrm{e}^{-\eta}, \tag{66}$$

which is invariant under the following infinitesimal gauge transformations,

$$\delta\rho = -\varepsilon\rho, \qquad \delta\theta = \varepsilon\theta, \qquad \delta\alpha = \varepsilon\alpha, \qquad \delta\beta = 0, \qquad \delta\eta = -\varepsilon, \tag{67}$$

where the ε is a time-dependent parameter. This reveals the time rescaling symmetry [3] in the nonrelativistic rotational fluid model.

5. Relativistic fluid mechanics

5.1. Introduction

The equations of motion of a perfect (dissipationless) relativistic fluid can be expressed in terms of a conserved and symmetric energy–momentum tensor $T_{\mu\nu}$, derived from Poincaré invariance by Noether's theorem. The general form of the energy–momentum tensor of a relativistic perfect fluid is [1, 14]:

$$T_{\mu\nu} = pg_{\mu\nu} + (\varepsilon + p)u_\mu u_\nu, \tag{68}$$

where p is the pressure, ε is the energy density and u^μ is the velocity 4-vector, which in natural units ($c = 1$) is a time-like unit vector: $u_\mu^2 = -1$. Local energy–momentum conservation is expressed by the vanishing of the four-divergence of the energy–momentum tensor

$$\partial^\mu T_{\mu\nu} = 0. \tag{69}$$

The conserved energy–momentum four-vector is then given in a laboratory inertial frame by

$$P_\mu = \int_{t=t_0} \mathrm{d}^3x\, T_{\mu 0}, \qquad \frac{\mathrm{d}P_\mu}{\mathrm{d}t} = 0. \tag{70}$$

In addition to the conservation of energy and momentum, the fluid density is conserved during ordinary flow as well. This is expressed by the vanishing divergence of the fluid density current j^μ:

$$\partial_\mu j^\nu = 0, \qquad j^\mu = \rho u^\mu, \tag{71}$$

where ρ represents the local fluid density in the local instantaneous rest frame; the normalization of the four-velocity then implies that the current satisfies

$$-j_\mu^2 = \rho^2 \geqslant 0. \tag{72}$$

Thus the local fluid density is defined in Lorentz invariant manner. In a space-plus-time formulation, equation (71) is seen to imply the equation of continuity

$$\partial.\vec{j} = \partial_t(\rho\gamma) + \nabla_i(\rho\gamma v^i) = 0, \qquad \gamma = (1 - v^2)^{-1/2}. \tag{73}$$

Because of the vanishing divergence, for general fluid flow the current has three independent components. A standard way to express this is to write the current in terms of three scalar potentials (θ, α, β); they are introduced as Lagrange multipliers combined in an auxiliary vector field a_μ, with the Clebsch decomposition

$$a_\mu = \partial_\mu\theta + \alpha\partial_\mu\beta. \tag{74}$$

In this formalism the component θ describes the pure potential flow, whilst α and β are necessary to include non-zero vorticity [21].

An alternative to the Clebsch decomposition, which is mathematically equivalent but has several advantages: it gives an insight into the construction of an infinite set of conserved currents, and it allows a straightforward supersymmetric generalization [13]. So, replacing the real Clebsch potentials (θ, α, β) by one real potential θ and one complex potential z, with its conjugate $\overline{z}$. In terms of these potentials a general Lagrange density for a relativistic fluid is given by the expression

$$\begin{aligned}\mathcal{L}(j^\mu, \theta, z, \overline{z}) &= -j^\mu a_\mu - f(\rho) \\ &= -j^\mu(\partial_\mu\theta + \mathrm{i}K_z\partial_\mu z - \mathrm{i}K_{\overline{z}}\partial_\mu\overline{z}) - f(\rho).\end{aligned} \tag{75}$$

Here $K(z, \overline{z})$ is a real function of the complex potentials, which we refer to as the Kähler potential; K_z and $K_{\overline{z}}$ are its partial derivatives w.r.t. z and $\overline{z}$, and f is a function of $\rho = \sqrt{-j^2}$ only.

5.2. Symplectic analysis

To perform the symplectic formalism the Lagrangian density is reduced to the first-order form, given by

$$\mathcal{L} = -\rho\dot{\theta} - \mathrm{i}\rho K_z\dot{z} + \mathrm{i}\rho K_{\overline{z}}\dot{\overline{z}} + j^i\partial_i\theta + \mathrm{i}j^iK_z\partial_i z - \mathrm{i}j^iK_{\overline{z}}\partial_i\overline{z} - f(\rho), \tag{76}$$

or

$$\mathcal{L}^{(0)} = -\theta\dot{\rho} - \mathrm{i}\rho K_z\dot{z} + \mathrm{i}\rho K_{\overline{z}}\dot{\overline{z}} - U^{(0)} \tag{77}$$

where the symplectic potentials are

$$U^{(0)} = -j^i\partial_i\theta - ij^iK_z\partial_i z + ij^iK_{\overline{z}}\partial_i\overline{z} + f(\rho). \tag{78}$$

The symplectic fields are $\xi_\gamma^{(0)} = (\rho, \theta, z, \overline{z})$ with the corresponding 1-form canonical momenta given by

$$A_\rho^{(0)} = \theta \qquad A_\theta^{(0)} = 0 \qquad A_z^{(0)} = -\mathrm{i}\rho K_z \qquad A_{\overline{z}}^{(0)} = \mathrm{i}\rho K_{\overline{z}}. \tag{79}$$

The zeroth-iterative symplectic matrix is

$$f^{(0)} = \begin{pmatrix} 0 & -1 & -\mathrm{i}K_z & \mathrm{i}K_{\overline{z}} \\ 1 & 0 & 0 & 0 \\ \mathrm{i}K_z & 0 & 0 & 2\mathrm{i}\rho K_{z\overline{z}} \\ -\mathrm{i}K_{\overline{z}} & 0 & -2\mathrm{i}\rho K_{z\overline{z}} & 0 \end{pmatrix} \delta(\vec{r} - \vec{r}\,'). \tag{80}$$

This is a nonsingular matrix whose inverse is

$$f^{(0)^{-1}} = \begin{pmatrix} 0 & 1 & 0_z & 0 \\ -1 & 0 & \dfrac{K_{\overline{z}}}{2\rho K_{z\overline{z}}} & \dfrac{K_z}{2\rho K_{z\overline{z}}} \\ 0 & -\dfrac{K_{\overline{z}}}{2\rho K_{z\overline{z}}} & 0 & \dfrac{\mathrm{i}}{2\rho K_{z\overline{z}}} \\ 0 & -\dfrac{K_z}{2\rho K_{z\overline{z}}} & -\dfrac{\mathrm{i}}{2\rho K_{z\overline{z}}} & 0 \end{pmatrix} \delta(\vec{r} - \vec{r}\,'). \tag{81}$$

The model is not a gauge invariant field theory. As settled by the symplectic formalism, the Dirac brackets among the phase-space fields are

$$\begin{aligned}
\{z(\vec{r},t),\bar{z}(\vec{r}\,',t)\}^* &= \frac{\mathrm{i}}{2\rho K_{z\bar{z}}}\delta(\vec{r}-\vec{r}\,'),\\
\{\theta(\vec{r},t),\rho(\vec{r}\,',t)\}^* &= \delta(\vec{r}-\vec{r}\,'),\\
\{z(\vec{r},t),\theta(\vec{r}\,',t)\}^* &= \frac{K_{\bar{z}}}{2\rho K_{z\bar{z}}}\delta(\vec{r}-\vec{r}\,'),\\
\{\bar{z}(\vec{r},t),\theta(\vec{r}\,',t)\}^* &= \frac{K_{z}}{2\rho K_{z\bar{z}}}\delta(\vec{r}-\vec{r}\,').
\end{aligned} \tag{82}$$

That completes the noinvariant analysis.

5.3. The WZ gauge model

In order to reformulate the model as a gauge invariant field theory, let us start with the first-order Lagrangian $\mathcal{L}^{(0)}$, equation (77), added with the arbitrary terms (Ψ, G), given by

$$\tilde{\mathcal{L}}^{(0)} = -\theta\dot{\rho} - \mathrm{i}\rho K_z\dot{z} + \mathrm{i}\rho K_{\bar{z}}\dot{\bar{z}} + \Psi\dot{\eta} - \tilde{U}^{(0)} \tag{83}$$

with

$$\tilde{U}^{(0)} = -j^i\partial_i\theta - \mathrm{i}j^iK_z\partial_iz + \mathrm{i}j^iK_{\bar{z}}\partial_i\bar{z} + f(\rho) + G, \tag{84}$$

where $\Psi \equiv \Psi(\rho,\theta,z,\bar{z})$ and $G \equiv G(\rho,\theta,z,\bar{z},\eta)$ are arbitrary functions to be determined. Now, the symplectic fields are $\tilde{\xi}^{(0)}_{\tilde{\gamma}} = (\rho,\theta,z,\bar{z},\eta)$ while the symplectic matrix is

$$\tilde{f}^{(0)} = \begin{pmatrix}
0 & \delta(\vec{r}-\vec{r}\,') & -\mathrm{i}K_z\delta(\vec{r}-\vec{r}\,') & \mathrm{i}K_{\bar{z}}\delta(\vec{r}-\vec{r}\,') & \dfrac{\delta\Psi_{\vec{r}\,'}}{\delta\rho(\vec{r})}\\
-\delta(\vec{r}-\vec{r}\,') & 0 & 0 & 0 & \dfrac{\delta\Psi_{\vec{r}\,'}}{\delta\theta(\vec{r})}\\
\mathrm{i}K_z\delta(\vec{r}-\vec{r}\,') & 0 & 0 & 2\mathrm{i}\rho K_{z\bar{z}}\delta(\vec{r}-\vec{r}\,') & \dfrac{\delta\Psi_{\vec{r}\,'}}{\delta z(\vec{r})}\\
-\mathrm{i}K_{\bar{z}}\delta(\vec{r}-\vec{r}\,') & 0 & -2\mathrm{i}\rho K_{z\bar{z}}\delta(\vec{r}-\vec{r}\,') & 0 & \dfrac{\delta\Psi_{\vec{r}\,'}}{\delta\bar{z}(\vec{r})}\\
-\dfrac{\delta\Psi_{\vec{r}}}{\delta\rho(\vec{r}\,')} & -\dfrac{\delta\Psi_{\vec{r}}}{\delta\theta(\vec{r}\,')} & -\dfrac{\delta\Psi_{\vec{r}}}{\delta z(\vec{r}\,')} & -\dfrac{\delta\Psi_{\vec{r}}}{\delta\bar{z}(\vec{r}\,')} & 0
\end{pmatrix}, \tag{85}$$

where $\Psi_{\vec{r}} \equiv \Psi(\vec{r})$

Now, let us explore a hidden symmetry associated with the zero mode

$$\tilde{\nu}^{(0)} = (0 \quad 1 \quad 0 \quad 0 \quad -1). \tag{86}$$

So, a set of differential equations is obtained as

$$\begin{aligned}
&\int \mathrm{d}\vec{r}\left(-\delta(\vec{r}-\vec{r}\,') + \frac{\delta\Psi(\vec{r})}{\delta\rho(\vec{r}\,')}\right) = 0, \qquad && \int \mathrm{d}\vec{r}\,\frac{\delta\Psi(\vec{r})}{\delta z(\vec{r}\,')} = 0,\\
&\int \mathrm{d}\vec{r}\,\frac{\delta\Psi(\vec{r})}{\delta\bar{z}(\vec{r}\,')} = 0, && \int \mathrm{d}\vec{r}\,\frac{\delta\Psi(\vec{r})}{\delta\theta(\vec{r}\,')} = 0.
\end{aligned} \tag{87}$$

After an integration process, Ψ is determined as being

$$\Psi(\vec{r}) = \rho(\vec{r}), \tag{88}$$

with the corresponding symplectic matrix

$$\tilde{f}^{(0)} = \begin{pmatrix} 0 & \delta(\vec{r}-\vec{r}\,') & -\mathrm{i}K_z & \mathrm{i}K_{\bar{z}} & \delta(\vec{r}-\vec{r}\,') \\ -\delta(\vec{r}-\vec{r}\,') & 0 & 0 & 0 & 0 \\ \mathrm{i}K_z & 0 & 0 & 2\mathrm{i}\rho K_{z\bar{z}} & 0 \\ -\mathrm{i}K_{\bar{z}} & 0 & -2\mathrm{i}\rho K_{z\bar{z}} & 0 & 0 \\ -\delta(\vec{r}-\vec{r}\,') & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{89}$$

This matrix is obviously singular and the first-order Lagrangian becomes

$$\tilde{\mathcal{L}}^{(0)} = -\theta\dot{\rho} - \mathrm{i}\rho K_z\dot{z} + \mathrm{i}\rho K_{\bar{z}}\dot{\bar{z}} + \rho\dot{\eta} - \tilde{U}^{(0)}. \tag{90}$$

Now, let us begin with the second step in order to reformulate the model as a WZ gauge invariant theory. The zero mode $\tilde{\nu}^{(0)}$ does not produce a constraint when contracted with the gradient of the symplectic potential, namely,

$$\int \mathrm{d}\vec{r}\; \tilde{\nu}^{(0)}(\vec{r})\frac{\delta\tilde{U}^{(0)}(\vec{r}\,')}{\delta\tilde{\xi}^{(0)}(\vec{r})} = 0; \tag{91}$$

instead, this produces a general differential equation that allows the computation of whole correction terms in the order of η enclosed into $G(\rho,\theta,z,\bar{z},\eta)$. For linear correction term in η, we have

$$\int \mathrm{d}\vec{r}\,' \left[-j^i\partial_i\delta(\vec{r}-\vec{r}\,') + \frac{\delta\mathcal{G}^{(1)}(\vec{r}\,')}{\delta\eta(\vec{r})}\right] = 0. \tag{92}$$

After straightforward calculation, the linear correction term in the order of η is obtained as

$$\mathcal{G}^{(1)} = -j^i\partial_i\eta. \tag{93}$$

Since this correction term has no dependence on θ, the remaining correction terms are null. As a consequence, the symplectic potential (83) becomes

$$\tilde{U}^{(0)} = -j^i\partial_i\theta - \mathrm{i}j^iK_z\partial_iz + \mathrm{i}j^iK_{\bar{z}}\partial_i\bar{z} + f(\rho) - j^i\partial_i\eta. \tag{94}$$

Hence, the gauge invariant first-order Lagrangian is written as

$$\tilde{\mathcal{L}}^{(0)} = -(\theta+\eta)\dot{\rho} - \mathrm{i}\rho K_z\dot{z} + \mathrm{i}\rho K_{\bar{z}}\dot{\bar{z}} - \tilde{U}^{(0)}. \tag{95}$$

where the symplectic potential is

$$\tilde{U}^{(0)} = -j^i\partial_i(\theta+\eta) - \mathrm{i}j^iK_z\partial_iz + \mathrm{i}j^iK_{\bar{z}}\partial_i\bar{z} + f(\rho). \tag{96}$$

By construction, the contraction of the zero mode ($\tilde{\nu}^{(0)}$) with the gradient of the symplectic potential above does not produce a new constraint; consequently, a WZ symmetry is disclosed.

The infinitesimal gauge transformations, which let the Hamiltonian invariant ($\tilde{U}^{(0)}$), are

$$\delta\rho = 0, \qquad \delta\theta = \varepsilon, \qquad \delta z = 0, \qquad \delta\bar{z} = 0, \qquad \delta\eta = -\varepsilon. \tag{97}$$

Consequently, the invariance under global translation of the velocity potential was lifted to a local invariance, likely in the nonrelativistic irrotational fluid. However, unlike the nonrelativistic rotational fluid model, the symmetry produced by the zero mode, equation (86), seems unique for a general Kähler potential, i.e., there is not a family of dynamically equivalent WZ symmetries or the possibility of existing local description of global extra symmetries [3]. But when this potential is null, the result obtained in the context of the nonrelativistic model (section 4) can be extended to the relativistic one.

6. Conclusions

In this paper, we investigate hidden symmetries on the nonrelativistic and relativistic rotational fluid model using the symplectic embedding formalism. In particular, we demonstrate that the nonrelativistic description of the rotational fluid mechanics also presents extra hidden symmetry, as its respective irrotational description [3, 12] and, also, we show that this model has a set of dynamically equivalent WZ symmetries. It is opportune to mention that we do not explore another WZ hidden symmetry on the model since it is a straight application of the embedding formalism with another choice for the zero mode (vide [12] for details). In the context of the relativistic rotational fluid mechanics, the global translation symmetry of the velocity potential was lifted to the local one. Further, we could argue that this is the unique gauge symmetry presented on the model and that the extra symmetries [3] present on the nonrelativistic fluid model are not present in the relativistic one, at least when the Käahler potential is not equal to zero. This is a strong condition inputed by the Käahler potential for the existence of some symmetry on the model.

It is well known that the dissipation exists in the real world, although sometimes extremely small [22]. Further, the physically meaningful theories are those that present at least an infinitesimal amount of dissipation. In order to shed some light on this problem, we are working on the investigation of the degeneracy present on the descriptions of inviscid hydrodynamics lifted with infinitesimal dissipation, where we believe it is possible to bring new insights into the issue.

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